JOURNAL OF APPROXIMATION THEORY **91**, 84–102 (1997)
ARTICLE NO. AT963089

Weighted Norm Inequalities for Fractional Integrals with an Application to Mean Convergence of Laguerre Series

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Communicated by D. Lubinsky

Received September 5, 1995; accepted in revised form October 15, 1996

We prove norm inequalities with exponential weights for the Riemann–Liouville fractional integral. As an application, we show for certain functions that their Laguerre expansions will converge in the L^p norm for some p outside the standard range of (4/3, 4).

1. INTRODUCTION

The Riemann–Liouville fractional integral with origin 0 is defined:

$$f_a(x)=\frac{1}{\Gamma(a)}\int_0^x f(t)(x-t)^{a-1}\,dt, \qquad a>0. \tag{1.1}$$

In [4], Hardy and Littlewood proved that if $p>1$, $l>1/p-1$, $x^{-l}f\in L^p(0,\infty)$, $0\leqslant m\leqslant a<1/p$, $a>0$ if $m=0$, and $q=1/(1/p+m-a)$, then

$$\int_0^\infty |x^{-(m+l)}f_a|^q\,dx\leqslant K\left(\int_0^\infty |x^{-l}f|^p\,dx\right)^{q/p}, \tag{1.2}$$

where $K=K(a,m,l,p)$ only. They also showed that (1.2) does not hold in general for other values of q.

The restriction $a<1/p$ in (1.2) can be relaxed to $a<m+1/p$, as we will see later.

Hardy and Littlewood's results sparked a great interest in the topic of norm inequalities for fractional integrals. Most recently, this interest has led to characterizations of weights u and w for which a fractional integral inequality

$$\int_0^\infty |f_a(x)|^q\,w(x)\,dx\leqslant K\left(\int_0^\infty |f(x)|^p\,u(x)\,dx\right)^{q/p}$$

0021-9045/97 $25.00

will hold for certain values of q and p. This type of characterization usually takes on one of the following forms: (i) to determine weights for which the norm inequality holds if $1<p<1/a$, $q=1/(1/p-a)$; (ii) to determine weights for which the norm inequality holds for $q=p$; or (iii) to characterize weights using certain classes of test functions related to the weights.

For examples of these types of results, see e.g. [8] and his extensive list of references. We note that while characterizations of the type (iii) are interesting, it is not necessarily easier to verify the characterization for a given weight than it is to prove the norm inequality directly.

In this paper we investigate fractional integral inequalities for "Laguerre" weights. For given values of p, we determine all values of q for which the inequality will hold and give examples to show that our range is best possible. We note that our weights are not in the general weight classes $A_{p,q}$ as discussed in e.g. [10]. Our interest in these weights is based on their applications to Laguerre expansions.

More specifically, we prove that if a, $b>0$, $p>1$, $m\geqslant 0$, $l>1/p-1$, $x^{-l}e^{-bx}f\in L^p(0,\infty)$, then

$$\int_0^\infty |x^{-(m+l)}e^{-bx}f_a|^q\,dx\leqslant K\left(\int_0^\infty |x^{-l}e^{-bx}f|^p\,dx\right)^{q/p}, \tag{1.3}$$

where $K=K(a, b, q, p, l, m)$ only, is true for $m=0$ if

$$0<a<\frac{1}{p}\qquad\text{and}\qquad p\leqslant q\leqslant\frac{1}{1/p-a}, \tag{1.4a}$$

$$a=\frac{1}{p}\qquad\text{and}\qquad p\leqslant q<\infty, \tag{1.4b}$$

or

$$a>\frac{1}{p}\qquad\text{and}\qquad p\leqslant q\leqslant\infty; \tag{1.4c}$$

and for $m>0$ if

$$0<a<m\qquad\text{and}\qquad \frac{1}{1/p+m}<q<\frac{1}{1/p+m-a}, \tag{1.5a}$$

$$m\leqslant a<m+\frac{1}{p}\qquad\text{and}\qquad \frac{1}{1/p+m}<q\leqslant\frac{1}{1/p+m-a}, \tag{1.5b}$$

or

$$a\geqslant m+\frac{1}{p}\qquad\text{and}\qquad \frac{1}{1/p+m}<q\leqslant\infty. \tag{1.5c}$$

We give examples to show that (1.3) is not generally true for q outside the intervals above.

Note that

$$\int_0^\infty |x^{-(m+l)} e^{-bx} f_a|^q \, dx \leqslant \int_0^\infty |x^{-(m+l)} (e^{-bt} f)_a|^q \, dx \tag{1.6}$$

so that the case $q = 1/(1/p + m - a)$, $0 \leqslant m \leqslant a < 1/p$ in (1.3) follows from (1.2) applied to the function $e^{-bx} f$.

We will apply (1.3) to mean convergence of Laguerre expansions.

For $\lambda > -1$ the Laguerre polynomials of index λ are defined to be the unique polynomials $L_k^\lambda(x)$, $k = 0, 1, \ldots$ which satisfy the orthogonality relation

$$\int_0^\infty L_k^\lambda(x) \, L_j^\lambda(x) \, e^{-x} x^\lambda \, dx = \frac{\Gamma(k+\lambda+1)}{k!} \, \delta_{k,j}, \tag{1.7}$$

and the requirement that each L_k^λ is of exact degree k with coefficient of x^k being $(-1)^k/k!$.

If $x^{\lambda/2} e^{-x/2} f \in L^p(0, \infty)$, the nth partial sum of its Laguerre expansion of index λ is defined by

$$S_n^\lambda(x) = \sum_{k=0}^{n} a_k L_k^\lambda(x), \tag{1.8}$$

where the Laguerre coefficients are given by

$$a_k = \frac{k!}{\Gamma(k+\lambda+1)} \int_0^\infty f(x) \, L_k^\lambda(x) \, x^\lambda e^{-x} \, dx. \tag{1.9}$$

Then a famous theorem of Askey and Wainger [1] says that if $\frac{4}{3} < p < 4$, then

$$\| x^{\lambda/2} e^{-x/2} (S_n^\lambda - f) \|_p \to 0 \qquad \text{as} \quad n \to \infty. \tag{1.10}$$

They also show that (1.10) is not generally true for $1 \leqslant p \leqslant \frac{4}{3}$ and $p \geqslant 4$.

The connection between Laguerre polynomials and fractional integration is given by the formula:

For $\beta > -1$, $a > 0$ [2]:

$$(t^\beta L_k^\beta)_a \, (x) = \frac{\Gamma(\beta+k+1)}{\Gamma(a+\beta+k+1)} \, x^{a+\beta} L_k^{a+\beta}(x). \tag{1.11}$$

Notice that

$$\frac{\Gamma(\beta+k+1)}{\Gamma(a+\beta+k+1)}=O(k^{-a}), \qquad k\to\infty, \tag{1.12}$$

so that if a function f has a Laguerre expansion of index β, and we consider the function $g(x)=x^{-(a+\beta)}(t^{\beta}f)_{a}(x)$, then, because the coefficients are affected in a favorable way, the expansion for g (of order $a+\beta$) should in some sense behave better than the expansion for f (of order β).

Our fractional integral inequalities interpret this phenomena in the context of mean convergence. We show that if $\lambda>\max(-1, -2/p')$, $1/p+1/p'=1$, $\frac{4}{3}<p<4$, $0<a<\lambda+2/p'$, if also

$$x^{(a-\lambda)/2}e^{-x/2}(t^{\lambda}f)^{(a)}(x)\in L^{p}(0, \infty), \tag{1.13}$$

where $(t^{\lambda}f)^{(a)}(x)$ is the ath fractional derivative of $x^{\lambda}f$, then there exist coefficients $c_0, c_1, \ldots$ so that

$$\left\| e^{-x/2}x^{\lambda/2}\left(\sum_{k=0}^{n} c_k L_k^{\lambda}-f\right)\right\|_q \to 0 \qquad \text{as} \quad n\to\infty \tag{1.14}$$

whenever

$$0<a<\frac{2}{p} \qquad \text{and} \qquad \frac{1}{\frac{1}{p}+\frac{a}{2}}<q\leqslant\frac{1}{\frac{1}{p}-\frac{a}{2}}; \tag{1.15a}$$

or

$$a\geqslant\frac{2}{p} \qquad \text{and} \qquad \frac{1}{\frac{1}{p}+\frac{a}{2}}<q\leqslant\infty. \tag{1.15b}$$

If $\lambda>-2/q'$, $1/q'+1/q=1$, we show that the sum in (1.14) coincides with S_n^{λ} defined in (1.8).

As an example, we will analyze the mean convergence of the Laguerre expansions of $f(x)=x^{\nu}$.

2. PRELIMINARIES

In this section we prove a lemma on which our proof of (1.3) is based. We will need the left handed Hardy–Littlewood maximal function

$$\mathscr{M}g(x) = \sup_{0<h\leqslant x} \frac{1}{h}\int_{x-h}^{x} |g(t)|\,dt \tag{2.1}$$

and the incomplete gamma function

$$\gamma(a, x) = \int_0^x t^{a-1}e^{-t}\,dt, \qquad a>0. \tag{2.2}$$

We note for future reference that $\gamma(a, x)$ is continuous, $\gamma(a, x)>0$ for all $x>0$ and

$$\lim_{x\to 0+} x^{-a}\gamma(a, x) = \frac{1}{a}, \tag{2.3a}$$

$$\lim_{x\to\infty} \gamma(a, x) = \Gamma(a). \tag{2.3b}$$

For notational simplicity, define

$$\mathscr{G}_{a,b}(x) = b^{-a}\gamma(a, bx) = \int_0^x t^{a-1}e^{-bt}\,dt. \tag{2.4}$$

We will use $\|\cdot\|_p$ to denote the $L^p(0,\infty)$ norm.

LEMMA 2.1. *If $b>0$, $p>1$, $0\leqslant\theta\leqslant 1$, $a>\theta/p$, $g\in L^p(0,\infty)$ and $x\geqslant 0$ we have*

$$e^{-bx}\int_0^x |g(t)|\,e^{bt}(x-t)^{a-1}\,dt \leqslant K\,\|g\|_p^{\theta}\,(\mathscr{M}g(x))^{1-\theta}\,\mathscr{G}_{a-\theta/p,\,b}(x), \tag{2.5}$$

where $K=K(a, b, \theta, p)$ only.

Proof. We may assume $0<\mathscr{M}g(x)<\infty$, $g\geqslant 0$. Define

$$G(t) = \int_0^t g(x-u)\,du. \tag{2.6}$$

Then clearly $G(t)$ is majorized by both $t\mathscr{M}g(x)$ and $t^{1/p'}\|g\|_p$ so that for $0\leqslant\theta\leqslant 1$ we have

$$G(t)\leqslant (t\mathscr{M}g(x))^{1-\theta}\,(t^{1/p'}\,\|g\|_p)^{\theta}=t^{1-\theta/p}\,\|g\|_p^{\theta}\,(\mathscr{M}g(x))^{1-\theta}, \tag{2.7}$$

where $1/p+1/p'=1$.

Writing $w(t)=e^{-bt}t^{a-1}$, we have

$$\begin{aligned} e^{-bx}\int_0^x g(t)\,e^{bt}(x-t)^{a-1}\,dt &= \int_0^x g(x-t)\,w(t)\,dt \\ &= G(x)w(x)-\int_0^x G(t)\,w'(t)\,dt \\ &\leqslant \|g\|_p^{\theta}(\mathscr{M}g(x))^{1-\theta}\left(x^{1-\theta/p}w(x)+\int_0^x t^{1-\theta/p}\,|w'(t)|\,dt\right). \end{aligned} \tag{2.8}$$

From (2.4), it is easy to see that

$$x^{1-\theta/p}w(x)=x^{a-\theta/p}e^{-bx}\leqslant\left(a-\frac{\theta}{p}\right)\mathscr{G}_{a-\theta/p,\,b}(x). \tag{2.9}$$

Furthermore, since $w'(t)=w(t)(a-1-bt)/t$ we have

$$\begin{aligned} \int_0^x t^{1-\theta/p}\,|w'(t)|\,dt &\leqslant |a-1|\int_0^x t^{-\theta/p}w(t)\,dt+b\int_0^x t^{1-\theta/p}w(t)\,dt \\ &= |a-1|\,\mathscr{G}_{a-\theta/p,\,b}(x)+b\mathscr{G}_{1+a-\theta/p,\,b}(x) \\ &\leqslant K\mathscr{G}_{a-\theta/p,\,b}(x), \end{aligned} \tag{2.10}$$

where the last inequality follows from (2.3a), (2.3b), and the remarks preceding those equations. Now, combining (2.9) and (2.10) with (2.8) completes the proof.

The following related formula is given in [5] for $p>1$, $0<a<1/p$,

$$f_a(x)\leqslant K\,\|f\|_p^{pa}\,(\mathscr{M}f(x))^{1-pa}, \tag{2.11}$$

where $K=K(a, p)$ only. It follows that for $g \geqslant 0$, $b>0$, $p>1$, $m \geqslant 0$ and $m \leqslant a<m+1/p$, $a>0$ if $m=0$, $K=K(a, b, p, m)$,

$$e^{-bx} \int_0^x g(t)\, e^{bt}(x-t)^{a-1}\, dt$$

$$\leqslant K x^m g_{a-m}(x) \leqslant K x^m \, \|g\|_p^{p(a-m)} \, (\mathscr{M} g(x))^{1-p(a-m)}, \qquad (2.12)$$

which for $m=0$ extends (2.5) to $\theta=pa$.

3. WEIGHTED INEQUALITIES

In applying equations (2.5) and (2.12) we will use the norm inequality

$$\|\mathscr{M} g\|_p \leqslant K \, \|g\|_p, \qquad p>1, \qquad (3.1)$$

where $K=K(p)$ only (see e.g. [4, Thm. 398]), Hölder's inequality,

$$\int |fg| \leqslant \|f\|_p \, \|g\|_{p'}, \qquad \frac{1}{p}+\frac{1}{p'}=1, \qquad (3.2)$$

and the limits (2.3a) and (2.3b).

We will first prove some preliminary versions of (1.3).

THEOREM 3.1. *If $p>1$, a, $b>0$ and $e^{-bx} f \in L^p(0, \infty)$, then*

$$\|e^{-bx} f_a\|_q \leqslant K \, \|e^{-bx} f\|_p, \qquad (3.3)$$

where $K=K(a, b, q, p)$ only, is true whenever

$$0<a<\frac{1}{p} \quad \text{and} \quad p \leqslant q \leqslant \frac{1}{1/p-a}, \qquad (3.4a)$$

$$a=\frac{1}{p} \quad \text{and} \quad p \leqslant q<\infty, \qquad (3.4b)$$

or

$$a>\frac{1}{p} \quad \text{and} \quad p \leqslant q \leqslant \infty. \qquad (3.4c)$$

Proof. We can assume $f \geqslant 0$. Let $g(x)=e^{-bx} f(x)$.

Assume first that $0<a<1/p$ and $p\leqslant q<1/(1/p-a)$ or that $a\geqslant 1/p$ and $p\leqslant q<\infty$. Then $a>1/p-1/q$, so by Lemma 2.1 with $\theta=1-p/q$ we have

$$\begin{aligned} e^{-bx}f_a(x) &= e^{-bx}(e^{bt}g)_a(x) \\ &\leqslant K\,\|g\|_p^{1-p/q}\,(\mathscr{M}g(x))^{p/q}\,\mathscr{G}_{a+1/q-1/p,\,b}(x). \end{aligned} \tag{3.5}$$

Therefore we obtain

$$\begin{aligned} \int_0^\infty (e^{-bx}f_a)^q\,dx &\leqslant K\,\|g\|_p^{q-p}\,\|\mathscr{G}_{a+1/q-1/p,\,b}\|_\infty^q \int_0^\infty (\mathscr{M}g)^p\,dx \\ &\leqslant K\,\|g\|_p^q = K\,\|e^{-bx}f\|_p^q. \end{aligned} \tag{3.6}$$

If instead, we have $a>1/p$, $q=\infty$, we have by Lemma 2.1 with $\theta=1$,

$$e^{-bx}f_a(x)\leqslant K\,\|g\|_p\,\mathscr{G}_{a-1/p,\,b}(x) \tag{3.7}$$

so that

$$\|e^{-bx}f_a\|_\infty\leqslant K\,\|g\|_p\,\|\mathscr{G}_{a-1/p,\,b}\|_\infty = K\,\|e^{-bx}f\|_p. \tag{3.8}$$

Finally, the case $0<a<1/p$, $q=1/(1/p-a)$ follows from (1.2) and (1.6).

THEOREM 3.2. *If* $p>1$, $a, b>0$, $m>0$ *and* $e^{-bx}f\in L^p(0,\infty)$ *then*

$$\|x^{-m}e^{-bx}f_a\|_q\leqslant K\,\|e^{-bx}f\|_p, \tag{3.9}$$

where $K=K(a, b, m, p, q)$ *only, is true if*

$$0<a<m \qquad \textit{and} \qquad \frac{1}{1/p+m}<q<\frac{1}{1/p+m-a}, \tag{3.10a}$$

$$m\leqslant a<m+\frac{1}{p} \qquad \textit{and} \qquad \frac{1}{1/p+m}<q\leqslant\frac{1}{1/p+m-a}, \tag{3.10b}$$

or

$$a\geqslant m+\frac{1}{p} \qquad \textit{and} \qquad \frac{1}{1/p+m}<q\leqslant\infty. \tag{3.10c}$$

Proof. We can assume $f\geqslant 0$. As in the previous theorem we let $g(x)=e^{-bx}f(x)$.

We first look for values of $q<p$ for which (3.9) is true. Let $r=p/q$. Let r' be its conjugate exponent: $r'=p/(p-q)$ if $p<\infty$, $r'=1$ if $p=\infty$. By Lemma 2.1 with $\theta=0$ we have

$$e^{-bx}f_a(x)\leqslant K\mathscr{M}g(x)\,\mathscr{G}_{a,\,b}(x) \tag{3.11}$$

so that

$$\int_0^\infty (x^{-m}e^{-bx}f_a(x))^q\,dx \leqslant K\int_0^\infty (x^{-m}\mathcal{M}g(x)\,\mathcal{G}_{a,b}(x))^q\,dx$$
$$\leqslant K\,\|(x^{-m}\mathcal{G}_{a,b})^q\|_{r'}\,\|(\mathcal{M}g)^q\|_r$$
$$= K\,\|\mathcal{M}g\|_p^q$$
$$\leqslant K\,\|g\|_p^q$$
$$= K\,\|e^{-bx}f\|_p^q \tag{3.12}$$

provided that (i) $(a-m)\,qr' > -1$ and (ii) $-mqr' < -1$. Simplifying, we see that (i) holds if and only if either $a-m \geqslant 1/p$ or both $a-m < 1/p$ and $q < 1/(1/p+m-a)$. Also, (ii) holds if and only if $q > 1/(1/p+m)$. (If $p=\infty$ we interpret $1/\infty = 0$.) This proves (3.9) if

$$0 < a < m+\frac{1}{p} \qquad \text{and} \qquad \frac{1}{1/p+m} < q < \min\left(p, \frac{1}{1/p+m-a}\right) \tag{3.13a}$$

or

$$a \geqslant m+\frac{1}{p} \qquad \text{and} \qquad \frac{1}{1/p+m} < q < p. \tag{3.13b}$$

Notice that $a<m$ if and only if $1/(1/p+m-a)<p$, so that (3.13a) and (3.13b) can be rewritten

$$0 < a < m \qquad \text{and} \qquad \frac{1}{1/p+m} < q < \frac{1}{1/p+m-a} \tag{3.14a}$$

or

$$a \geqslant m \qquad \text{and} \qquad \frac{1}{1/p+m} < q < p. \tag{3.14b}$$

Therefore, (3.9) is proved under condition (3.10a).

If $m \leqslant a < m+1/p$, (3.9) under condition (3.10b) follows from (2.12) and (3.14b) using the Riesz–Thorin interpolation theorem (see e.g. [9]).

Finally if $a \geqslant m+1/p$, we have from Lemma 2.1 with $\theta = 1$

$$e^{-bx}f_a(x) \leqslant K\,\|g\|_p\,\mathcal{G}_{a-1/p,\,b}(x), \tag{3.15}$$

so that

$$\|x^{-m}e^{-bx}f_a\|_\infty \leqslant K\,\|g\|_p\,\|x^{-m}\mathcal{G}_{a-1/p,\,b}\|_\infty = K\,\|e^{-bx}f\|_p. \tag{3.16}$$

Therefore, by (3.14b), (3.16) and interpolation, we have proved (3.9) under condition (3.10c).

This concludes the proof of the theorem.

To prove the most general case of our inequality we need the following:

LEMMA 3.3. *Assume* $0 \leqslant k < 1 - 1/p$, $p > 1$, $g \in L^p(0, \infty)$, $d > 0$. *Define*

$$T_k g(x) = \frac{1}{x^{1-k}} \int_0^x t^{-k} g(t)\, dt. \tag{3.17}$$

Then

$$\|x^c e^{-dx} T_k g\|_q \leqslant K \|g\|_p, \tag{3.18}$$

where $K = K(c, d, k, p, q)$ *only, is true whenever*

$$c < 0 \qquad \text{and} \qquad 0 < q < \frac{1}{1/p - c}, \tag{3.19a}$$

$$0 \leqslant c < \frac{1}{p} \qquad \text{and} \qquad 0 < q \leqslant \frac{1}{1/p - c}, \tag{3.19b}$$

or

$$c \geqslant \frac{1}{p} \qquad \text{and} \qquad 0 < q \leqslant \infty. \tag{3.19c}$$

Proof. It is well-known that for $p > 1$ (see e.g. [4, p. 245, Eq. (9.9.8)]),

$$\|T_k f\|_p \leqslant K \|f\|_p, \qquad 0 \leqslant k < 1 - \frac{1}{p}. \tag{3.20}$$

If $q < p$, let $r = p/q$, and let r' be its conjugate exponent. Then

$$\begin{aligned} \|x^c e^{-dx} T_k g\|_q^q &\leqslant \|(x^c e^{-dx})^q\|_{r'} \|(T_k g)^q\|_r \\ &= \|(x^c e^{-dx})^q\|_{r'} \|T_k g\|_p^q \leqslant K \|g\|_p^q \end{aligned} \tag{3.21}$$

provided $cqr' > -1$, or equivalently, $c > 1/p - 1/q$. This is true if and only if $c \geqslant 1/p$ and $q < \infty$ or $c < 1/p$ and $q < 1/(1/p - c)$. If $c < 0$, then $1/(1/p - c) < p$, so (3.18) under condition (3.19a) is proved.

From now on we assume $c \geqslant 0$. So far in this case we have proved (3.18) for $0 < q < p$. Note that by Hölder's inequality

$$T_k g(x) \leqslant K \|g\|_p x^{-1/p}. \tag{3.22}$$

Therefore, if $p \leqslant q < \infty$, then

$$\int_0^\infty (x^c e^{-dx} T_k g)^q \, dx = \int_0^\infty (x^c e^{-dx})^q (T_k g)^{q-p} (T_k g)^p \, dx$$

$$\leqslant K \, \|g\|_p^{q-p} \int_0^\infty (x^{c-1/p+1/q} e^{-dx})^q (T_k g)^p \, dx$$

$$\leqslant K \, \|g\|_p^{q-p} \, \|x^{c-1/p+1/q} e^{-dx}\|_\infty^q \, \|T_k g\|_p^p$$

$$\leqslant K \, \|g\|_p^q \tag{3.23}$$

provided $c - 1/p + 1/q \geqslant 0$. This is true provided $c \geqslant 1/p$ or both $c < 1/p$ and $q \leqslant 1/(1/p - c)$. This proves (3.18) under condition (3.19b) and most of (3.19c).

Finally, if $q = \infty$, $c \geqslant 1/p$, then

$$\|x^c e^{-dx} T_k g\|_\infty \leqslant K \, \|g\|_p \, \|x^{c-1/p} e^{-dx}\|_\infty \leqslant K \, \|g\|_p. \tag{3.24}$$

This concludes the proof of the lemma.

THEOREM 3.4. *If* $a, b > 0$, $p > 1$, $m \geqslant 0$, $l > 1/p - 1$, $x^{-l} e^{-bx} f \in L^p(0, \infty)$, *then*

$$\int_0^\infty |x^{-(m+l)} e^{-bx} f_a|^q \, dx \leqslant K \left(\int_0^\infty |x^{-l} e^{-bx} f|^p \, dx \right)^{q/p}, \tag{3.25}$$

where $K = K(a, b, q, p, l, m)$ *only, is true for* $m = 0$ *if*

$$0 < a < \frac{1}{p} \qquad \textit{and} \qquad p \leqslant q \leqslant \frac{1}{1/p - a}, \tag{3.26a}$$

$$a = \frac{1}{p} \qquad \textit{and} \qquad p \leqslant q < \infty, \tag{3.26b}$$

or

$$a > \frac{1}{p} \qquad \textit{and} \qquad p \leqslant q \leqslant \infty, \tag{3.26c}$$

and for $m > 0$ *if*

$$0 < a < m \qquad \textit{and} \qquad \frac{1}{1/p + m} < q < \frac{1}{1/p + m - a}, \tag{3.27a}$$

$$m \leqslant a < m + \frac{1}{p} \qquad \textit{and} \qquad \frac{1}{1/p + m} < q \leqslant \frac{1}{1/p + m - a}, \tag{3.27b}$$

or

$$a \geqslant m + \frac{1}{p} \qquad \text{and} \qquad \frac{1}{1/p + m} < q \leqslant \infty. \tag{3.27c}$$

Furthermore, the norm inequality (3.25) *will not generally hold for q outside the intervals given above.*

Proof. We may assume $f \geqslant 0$.

Case 1. If $l = 0$, this is Theorems 3.1 and 3.2.

Case 2. If $l > 0$, then

$$x^{-(m+l)} e^{-bx} f_a(x) \leqslant x^{-m} e^{-bx} (t^{-l} f)_\alpha (x), \tag{3.28}$$

so Case 2 follows from Case 1 applied to $x^{-l} f$.

Case 3. If $1/p - 1 < l < 0$, let $g(x) = x^{-l} e^{-bx} f(x)$, $k = -l$. Observe that

$$\begin{aligned} &\Gamma(a)\, x^{-(m+l)} e^{-bx} f_a(x) \\ &\quad = x^{-(m+l)} e^{-bx} \left(\int_0^{x/2} f(t)(x-t)^{a-1}\, dt + \int_{x/2}^{x} f(t)(x-t)^{a-1}\, dt \right) \\ &\quad \leqslant K x^{-(m+l)} e^{-bx} \left(e^{bx/2} x^{a-1} \int_0^x t^l g(t)\, dt + x^l \int_0^x t^{-l} f(t)(x-t)^{a-1}\, dt \right) \\ &\quad = K(x^{a-m} e^{-bx/2} T_k\, g(x) + x^{-m} e^{-bx} (t^{-l} f)_a (x)). \end{aligned} \tag{3.29}$$

Therefore, Case 3 follows from Lemma 3.3 and Case 1.

Example. We first show that the norm inequality (3.25) cannot hold for $q \geqslant 1/(1/p + m - a)$, $0 < a < m$, or for $q > 1/(1/p + m - a)$, $0 \leqslant m \leqslant a < m + 1/p$, $a > 0$ if $m = 0$.

Let

$$f(x) = f(\varepsilon, x) = x^{l - 1/p + \varepsilon}, \qquad \varepsilon > 0. \tag{3.30}$$

Then $x^{-l} e^{-bx} f \in L^p(0, \infty)$ for all $\varepsilon > 0$. Therefore, since [2]

$$f_a(x) = \frac{\Gamma(l - 1/p + \varepsilon + 1)}{\Gamma(l - 1/p + \varepsilon + a + 1)}\, x^{l - 1/p + \varepsilon + a} \tag{3.31}$$

and since $\Gamma(l-1/p+\varepsilon+1)/\Gamma(l-1/p+\varepsilon+a+1)$ is bounded as $\varepsilon \to 0$, if the norm inequality (3.25) holds, then there exists a constant K independent of ε such that

$$\|x^{-m-1/p+\varepsilon+a}e^{-bx}\|_q \leqslant K\,\|x^{-1/p+\varepsilon}e^{-bx}\|_p. \tag{3.32}$$

If $q>1/(1/p+m-a)$, $0<a<m+1/p$, then $(a-m-1/p)\,q<-1$, so if $\varepsilon>0$ is sufficiently small the L^q norm in (3.32) will diverge.

If $q=1/(1/p+m-a)$, $m>0$, $0<a<m$, we have to analyze the behavior of the norms above as $\varepsilon \to 0$. We can calculate the norms exactly:

$$\|x^{-m-1/p+a+\varepsilon}e^{-bx}\|_q = \|x^{-1/q+\varepsilon}e^{-bx}\|_q = \frac{1}{(bq)^{\varepsilon}}\,(\Gamma(q\varepsilon))^{1/q}, \tag{3.33}$$

$$\|x^{-1/p+\varepsilon}e^{-bx}\|_p = \frac{1}{(bp)^{\varepsilon}}\,(\Gamma(p\varepsilon))^{1/p}. \tag{3.34}$$

Therefore, since $\Gamma(z)$ has a simple pole at $z=0$, if (3.32) holds, then

$$\limsup_{\varepsilon \to 0+} \varepsilon^{-1/q+1/p} < \infty. \tag{3.35}$$

This contradicts $-1/q+1/p=a-m<0$.

We show next that the norm inequality cannot hold for $q<p$ if $m=0$, or for $q \leqslant 1/(1/p+m)$ if $m>0$. Let

$$f(x)=f(x,\varepsilon)=x^l e^{bx}(x+1)^{-1/p-\varepsilon}, \qquad \varepsilon>0. \tag{3.36}$$

Then $x^{-l}e^{-bx}f \in L^p(0,\infty)$ for all $\varepsilon>0$, and we have

$$\begin{aligned}
&\Gamma(a)\, f_a(x)\\
&\quad=\int_0^x t^l e^{bt}(t+1)^{-1/p-\varepsilon}\,(x-t)^{a-1}\,dt\\
&\quad\geqslant (x+1)^{-1/p-\varepsilon}\int_0^x t^l e^{bt}(x-t)^{a-1}\,dt\\
&\quad=(x+1)^{-1/p-\varepsilon}\,x^{a+l}\int_0^1 e^{bxt}t^l(1-t)^{a-1}\,dt\\
&\quad=\frac{\Gamma(l+1)\,\Gamma(a)}{\Gamma(a+l+1)}\,(x+1)^{-1/p-\varepsilon}\,x^{a+l}\,{}_1F_1(l+1;a+l+1;bx),
\end{aligned} \tag{3.37}$$

where ${}_1F_1$ is a confluent hypergeometric function. Above we used the integral representation ([6], p. 115)

$$
{}_1F_1(a; c; z) = \frac{\Gamma(c)}{\Gamma(a)\,\Gamma(c-a)} \int_0^1 e^{zt}\, t^{a-1}(1-t)^{c-a-1}\, dt
$$

$$
\operatorname{Re}(c) > \operatorname{Re}(a) > 0. \tag{3.38}
$$

From the asymptotic formula ([6], p. 128)

$$
{}_1F_1(a; c; z) \sim \frac{\Gamma(c)}{\Gamma(a)}\, e^{-z} z^{a-c}, \qquad \operatorname{Re}(z) \to \infty, \tag{3.39}
$$

and the fact that ${}_1F_1(a; c; x) > 0$ for all $x > 0$, it follows that there is a constant C such that

$$
x^{-l} e^{-bx} f_a(x) \geqslant C x^{-1/p-\varepsilon}, \qquad x \geqslant 1. \tag{3.40}
$$

Therefore we have

$$
\|x^{-(m+l)} e^{-bx} f_a\|_q \geqslant C \left(\int_1^\infty (x^{-m-1/p-\varepsilon})^q\, dx \right)^{1/q}, \tag{3.41}
$$

so if the norm inequality (3.25) holds then we must have

$$
\left(\int_1^\infty (x^{-m-1/p-\varepsilon})^q\, dx \right)^{1/q} \leqslant K\, \|(x+1)^{-1/p-\varepsilon}\|_p, \tag{3.42}
$$

with K independent of ε.

Observe now that if $q < 1/(1/p+m)$, $m \geqslant 0$, then $(-m-1/p)\, q > -1$. Therefore if $\varepsilon > 0$ is sufficiently small, then $(-m-1/p-\varepsilon)\, q > -1$ and the integral in (3.42) on the left will diverge.

If $q = 1/(1/p+m)$, $m > 0$, then we must analyze the behavior of the norms in (3.42) as $\varepsilon \to 0+$. Again we can calculate the norms exactly

$$
\left(\int_1^\infty (x^{-m-1/p-\varepsilon})^q\, dx \right)^{1/q} = \left(\int_1^\infty (x^{-1/q-\varepsilon})^q\, dx \right)^{1/q} = \left(\frac{1}{q\varepsilon} \right)^{1/q}, \tag{3.43}
$$

$$
\|(x+1)^{-1/p-\varepsilon}\|_p = \left(\frac{1}{p\varepsilon} \right)^{1/p}. \tag{3.44}
$$

Therefore if the norm inequality holds, then

$$
\limsup_{\varepsilon \to 0+} \varepsilon^{-1/q+1/p} < \infty; \tag{3.45}
$$

this contradicts $-1/q + 1/p = -m < 0$.

Finally, we consider the case $a=1/p$, $m=0$, $q=\infty$. First note that the inequality

$$\|x^{-l}e^{-bx}f_a\|_\infty \leqslant K\, \|x^{-l}e^{-bx}f\|_p \tag{3.46}$$

for all f such that $x^{-l}e^{-bx}f \in L^p(0,\infty)$ is equivalent to the inequality

$$\|x^{-l}e^{-bx}(t^l e^{bt}g)_a\|_\infty \leqslant K\, \|g\|_p \tag{3.47}$$

for all $g \in L^p(0,\infty)$. If we consider only g supported in the interval $[1,2]$, (3.47) implies that

$$\int_1^x g(t)(x-t)^{-1/p'}\,dt \tag{3.48}$$

is a bounded function of $x \in [1,2]$ for all $g \in L^p(1,2)$. But this is impossible by the argument in [3, p. 577].

For applications, we will need the following specialization of Theorem 3.4:

COROLLARY 3.5. *If* $p>1$, $x^{(a-\lambda)/2}e^{-x/2}f(x) \in L^p(0,\infty)$, $0<a<\lambda+2/p'$, $1/p+1/p'=1$, *we have*

$$\|x^{-\lambda/2}e^{-x/2}f_a\|_q \leqslant K\, \|x^{(a-\lambda)/2}e^{-x/2}f\|_p, \tag{3.49}$$

where $K=K(a,\lambda,p,q)$ *only, whenever*

$$0<a<\frac{2}{p} \qquad \text{and} \qquad \frac{1}{\frac{1}{p}+\frac{a}{2}}<q\leqslant\frac{1}{\frac{1}{p}-\frac{a}{2}} \tag{3.50a}$$

or

$$a\geqslant\frac{2}{p} \qquad \text{and} \qquad \frac{1}{\frac{1}{p}+\frac{a}{2}}<q\leqslant\infty. \tag{3.50b}$$

Proof. Take $m=a/2$, $l=(\lambda-a)/2$, $b=1/2$ in Theorem 3.4.

4. APPLICATION TO MEAN CONVERGENCE

Recall that if m is a natural number, and $m-1<a<m$, then the ath (fractional) derivative is defined

$$f^{(a)}(x)=\frac{d^m}{dx^m}\,f_{m-a}(x). \tag{4.1}$$

If f has an mth derivative, then its derivative $f^{(m)}(x)$ coincides with $\lim_{a\to m-} f^{(a)}(x)$. Furthermore, if $a \leqslant m$, then

$$(f^{(a)})_a(x) = f(x). \tag{4.2}$$

For information on fractional differentiation and integration, see e.g. [7].

THEOREM 4.1. *Suppose* $\lambda > \max(-1, -2/p')$, $1/p + 1/p' = 1$, $4/3 < p < 4$ *and* $0 < a < \lambda + 2/p'$. *Suppose also that* $x^\lambda f$ *has* m *derivatives if* $m - 1 < a \leqslant m$, *and suppose*

$$x^{(a-\lambda)/2} e^{-x/2} (t^\lambda f)^{(a)}(x) \in L^p(0, \infty). \tag{4.3}$$

Then there exist constants $c_0, c_1, \ldots$ *such that*

$$\left\| e^{-x/2} x^{\lambda/2} \left(\sum_{k=0}^{n} c_k L_k^\lambda - f \right) \right\|_q \to 0 \qquad \text{as} \quad n \to \infty, \tag{4.4}$$

provided that

$$0 < a < \frac{2}{p} \qquad \text{and} \qquad \frac{1}{\frac{1}{p} + \frac{a}{2}} < q \leqslant \frac{1}{\frac{1}{p} - \frac{a}{2}} \tag{4.5a}$$

or

$$a \geqslant \frac{2}{p} \qquad \text{and} \qquad \frac{1}{\frac{1}{p} + \frac{a}{2}} < q \leqslant \infty. \tag{4.5b}$$

Proof. Let $h(x) = x^{a-\lambda} g(x)$ where $g(x) = (t^\lambda f)^{(a)}(x)$. Then $x^{(\lambda-a)/2} e^{-x/2} h \in L^p(0, \infty)$. Set

$$T_n(x) = \sum_{k=0}^{n} b_k L_k^{\lambda-a}(x), \tag{4.6}$$

where

$$b_k = \frac{k!}{\Gamma(k+\lambda-a+1)} \int_0^\infty h(x)\, L_k^{\lambda-a}(x)\, x^{\lambda-a} e^{-x}\, dx. \tag{4.7}$$

By the Askey–Wainger theorem,

$$\| x^{(\lambda-a)/2} e^{-x/2} (T_n - h) \|_p \to 0 \qquad \text{as} \quad n \to \infty. \tag{4.8}$$

In other words,

$$\|x^{(a-\lambda)/2}e^{-x/2}(x^{\lambda-a}T_n-g)\|_p \to 0 \qquad \text{as} \quad n\to\infty. \tag{4.9}$$

Therefore by Corollary 3.5,

$$\|x^{-\lambda/2}e^{-x/2}(x^{\lambda-a}T_n-g)_a\|_q \to 0 \qquad \text{as} \quad n\to\infty \tag{4.10}$$

for all q, a satisfying (4.5a) or (4.5b).

But $g_a(x)=x^\lambda f(x)$ and

$$(t^{\lambda-a}T_n)_a(x)=\sum_{k=0}^{n} b_k \frac{\Gamma(\lambda-a+k+1)}{\Gamma(\lambda+k+1)} x^\lambda L_k^\lambda(x). \tag{4.11}$$

Therefore

$$\left\| x^{\lambda/2}e^{-x/2}\left(\sum_{k=0}^{n} b_k \frac{\Gamma(\lambda-a+k+1)}{\Gamma(\lambda+k+1)} L_k^\lambda - f\right)\right\|_q \to 0 \qquad \text{as} \quad n\to\infty. \tag{4.12}$$

This finishes the proof.

Remark 4.2. If in the above theorem we have $\lambda>-2/q'$, where $1/q+1/q'=1$, then the c_k in (4.12) coincide with the Laguerre coefficients defined in (1.9). Denote

$$\tilde{T}_n(x)=\sum_{k=0}^{n} b_k \frac{\Gamma(\lambda-a+k+1)}{\Gamma(\lambda+k+1)} L_k^\lambda(x). \tag{4.13}$$

Let a_k denote the Laguerre coefficients of f. Fix k, and let $n\geqslant k$. Then

$$\begin{aligned}
a_k &= \frac{k!}{\Gamma(k+\lambda+1)}\int_0^\infty f(x)\, L_k^\lambda(x)\, x^\lambda e^{-x}\,dx \\
&= \frac{k!}{\Gamma(k+\lambda+1)}\int_0^\infty (f(x)-\tilde{T}_n)\, L_k^\lambda(x)\, x^\lambda e^{-x}\,dx \\
&\quad + \frac{k!}{\Gamma(k+\lambda+1)}\int_0^\infty \tilde{T}_n(x)\, L_k^\lambda(x)\, x^\lambda e^{-x}\,dx \\
&= \frac{k!}{\Gamma(k+\lambda+1)}\int_0^\infty (f(x)-\tilde{T}_n)\, L_k^\lambda(x)\, x^\lambda e^{-x}\,dx \\
&\quad + b_k \frac{\Gamma(\lambda-a+k+1)}{\Gamma(\lambda+k+1)}.
\end{aligned} \tag{4.14}$$

By Hölder's inequality, we have

$$\left| \int_0^\infty (f(x) - \tilde{T}_n)\, L_k^\lambda(x)\, x^\lambda e^{-x}\, dx \right|$$
$$\leqslant \| x^{\lambda/2} e^{-x/2} (f - \tilde{T}_n) \|_q \, \| x^{\lambda/2} e^{-x} L_k^\lambda \|_{q'} \to 0 \tag{4.15}$$

as $n \to \infty$. Therefore by (4.14) and (4.15) we obtain

$$a_k = b_k \frac{\Gamma(\lambda - a + k + 1)}{\Gamma(\lambda + k + 1)}. \tag{4.16}$$

The following example illustrates the theorem:

EXAMPLE 4.3. Suppose $v \geqslant 1/p' - 1/p$, $1/p + 1/p' = 1$, $\lambda > \max(-1, -2/p')$, $4/3 < p < 4$. Let $f(x) = x^v$, and define S_n^λ as in (1.8). Then

$$\| x^{\lambda/2} e^{-x/2} (S_n^\lambda - f) \|_q \to 0 \qquad \text{as} \quad n \to \infty \tag{4.17}$$

provided one of the following hold:

$$-\frac{2}{p'} < \lambda < 2\left(\frac{1}{p} - \frac{1}{p'}\right), \qquad \frac{2}{2+\lambda} < q < \frac{2}{2(1/p - 1/p') - \lambda}, \tag{4.18a}$$

$$\lambda = 2\left(\frac{1}{p} - \frac{1}{p'}\right), \qquad \frac{2}{2+\lambda} < q < \infty, \tag{4.18b}$$

$$\lambda > 2\left(\frac{1}{p} - \frac{1}{p'}\right), \qquad \frac{2}{2+\lambda} < q \leqslant \infty. \tag{4.18c}$$

Proof. Let $a = \lambda + 2/p' - \varepsilon$, where $\varepsilon > 0$ is sufficiently small so that $a > 0$. Define

$$g(x) = g(\varepsilon, x) = \frac{\Gamma(v + \lambda + 1)}{\Gamma(v + \lambda - a + 1)}\, x^{v + \lambda - a}. \tag{4.19}$$

We have $0 < a < \lambda + 2/p'$ and $x^{(a - \lambda)/2} e^{-x/2} g \in L^p(0, \infty)$. Observe that $g(x) = (t^\lambda f)^{(a)}(x)$. Therefore by Theorem 4.1, we have coefficients $c_0, c_1, ...,$ with

$$\left\| x^{\lambda/2} e^{-x/2} \left(\sum_{k=0}^{n} c_k L_k^\lambda - f \right) \right\|_q \to 0 \qquad \text{as} \quad n \to \infty \tag{4.20}$$

provided one of the following hold:

$$0<\lambda+\frac{2}{p'}-\varepsilon<\frac{2}{p}, \qquad \frac{1}{\frac{\lambda}{2}+1-\frac{\varepsilon}{2}}<q\leqslant\frac{1}{\frac{1}{p}-\frac{1}{p'}-\frac{\lambda}{2}+\frac{\varepsilon}{2}}, \tag{4.21a}$$

$$\lambda+\frac{2}{p'}-\varepsilon\geqslant\frac{2}{p}, \qquad \frac{1}{\frac{\lambda}{2}+1-\frac{\varepsilon}{2}}<q\leqslant\infty. \tag{4.21b}$$

Letting $\varepsilon\to 0$ will finish the proof: For example, if λ satisfies $-2/p'<\lambda<2(1/p-1/p')$, then there exists $\varepsilon_\lambda>0$ so that if $0<\varepsilon<\varepsilon_\lambda$, then $0<\lambda+2/p'-\varepsilon<2/p$. Therefore (4.17) will hold for

$$\frac{1}{\frac{\lambda}{2}+1-\frac{\varepsilon}{2}}<q\leqslant\frac{1}{\frac{1}{p}-\frac{1}{p'}-\frac{\lambda}{2}+\frac{\varepsilon}{2}}, \qquad 0<\varepsilon<\varepsilon_\lambda. \tag{4.22}$$

Now by letting $\varepsilon\to 0$ we obtain (4.18a).

The other two cases can be argued similarly.

Finally note that if $q>2/(2+\lambda)$, then $\lambda>-2/q'$, so according to Remark 4.2 the series in (4.20) is indeed the Laguerre expansion S_n^λ.

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